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**A METHOD TO ESTIMATE THE MECHANICAL PROPERTIES OF A SOLID
MATERIAL SUBJECTED TO ISONIFICATION**

TO ALL WHOM IT MAY CONCERN:

BE IT KNOWN THAT ANDREW J. HULL, employee of the United States Government, citizen of the United States of America, and resident of Newport, County of Newport, State of Rhode Island has invented certain new and useful improvements entitled as set forth above of which the following is a specification:

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3 **A METHOD TO ESTIMATE THE MECHANICAL PROPERTIES OF A SOLID**
4 **MATERIAL SUBJECTED TO ISONIFICATION**

5
6 **STATEMENT OF GOVERNMENT INTEREST**

7 The invention described herein may be manufactured and used
8 by or for the Government of the United States of America for
9 governmental purposes without the payment of any royalties
10 thereon or therefor.

11
12 **BACKGROUND OF THE INVENTION**

13 **(1) Field of the Invention**

14 The present invention relates to a method for measuring
15 mechanical characteristics of materials. More particularly,
16 this invention provides a method which uses transfer functions
17 obtained by insonifying the material at different angles. Once
18 obtained, the transfer functions are manipulated to yield closed
19 form values of dilatational and shear wavespeeds. The wavespeeds
20 are combined to determine complex Lamé constants, complex
21 Young's modulus, complex shear modulus, and complex Poisson's
22 ratio for the material.

(2) Description of the Prior Art

Measuring the mechanical properties of slab-shaped (i.e., plates) materials are important in that these parameters significantly contribute to the static and dynamic response of structures built with such materials. Resonant techniques have been used to identify and measure longitudinal properties for many years (See D.M. Norris, Jr., and W.C. Young, "Complex Modulus Measurements by Longitudinal Vibration Testing," *Experimental Mechanics*, Volume 10, 1970, pp. 93-96; W.M. Madigosky and G.F. Lee, "Improved Resonance Technique for Materials Characterization," *Journal of the Acoustical Society of America*, Volume 73, Number 4, 1983, pp. 1374-1377; S.L. Garrett, "Resonant Acoustic Determination of Elastic Moduli," *Journal of the Acoustical Society of America*, Volume 88, Number 1, 1990, pp. 210-220; G.F. Lee and B. Hartmann, U.S. Patent Number 5,363,701; G.W. Rhodes, A. Migliori, and R.D. Dixon, U.S. Patent Number 5,495,763; and R.F. Gibson and E.O. Ayorinde, U.S. Patent Number 5,533,399).

These methods are based on comparing the measured eigenvalues of a structure to predicted eigenvalues from a model of the same structure. The model of the structure must have well-defined (typically closed form) eigenvalues for these methods to work. Additionally, resonant techniques only allow measurements at resonant frequencies. Most of these methods

1 typically do not measure shear wavespeeds (or modulus) and do
2 not have the ability to estimate Poisson's ratio.

3 Comparison of analytical models to measured frequency
4 response functions is another method used to estimate stiffness
5 and loss parameters of a structure (See B.J. Dobson, "A
6 Straight-Line Technique for Extracting Modal Properties From
7 Frequency Response Data," *Mechanical Systems and Signal*
8 *Processing*, Volume 1, 1987, pp. 29-40; T. Pritz, "Transfer
9 Function Method for Investigating the Complex Modulus of
10 Acoustic Materials: Rod-Like Specimen," *Journal of Sound and*
11 *Vibration*, Volume 81, 1982, pp. 359-376; W.M. Madigosky and G.F.
12 Lee, U.S. Patent Number 4,352,292; and W.M. Madigosky and G.F.
13 Lee, U.S. Patent Number 4,418,573). When the analytical model
14 agrees with one or more frequency response functions, the
15 parameters used to calculate the analytical model are considered
16 accurate. If the analytical model is formulated using a
17 numerical method, a comparison of the model to the data can be
18 difficult due to dispersion properties of the materials.

19 Another method to measure stiffness and loss is to deform
20 the material and measure the resistance to the indentation (See
21 W.M. Madigosky, U.S. Patent Number 5,365,457). However, this
22 method can physically damage the specimen if the deformation
23 causes the sample to enter the plastic region of deformation.

1 Others methods have used insonification as a means to
2 determine defects in composite laminate materials (See D.E.
3 Chimenti and Y. Bar-Cohen, U.S. Patent Number 4,674,334).
4 However, these methods do not measure material properties.

5 A method does exist to measure shear wave velocity and
6 Poisson's ratio in the earth using boreholes and seismic
7 receivers (See J.D. Ingram and O.Y. Liu, U.S. Patent Number
8 4,633,449). However, this method needs a large volume of
9 material and is not applicable to slab-shaped samples.
10 Additionally, it needs a borehole in the volume at some
11 location.

12 In view of the above, there is a need for a method to
13 measure complex frequency-dependent dilatational and shear
14 wavespeeds of materials subject to insonification. Once the
15 wavespeeds are identified, the complex frequency-dependent
16 Young's and shear moduli and complex frequency-dependent
17 Poisson's ratio can also be measured (or estimated).

18 19 **SUMMARY OF THE INVENTION**

20 Accordingly, it is a general purpose and primary object of
21 the present invention to provide a method to measure (or
22 estimate) the complex frequency-dependent dilatational and shear
23 wavespeeds of a slab of material subjected to insonification.

1 It is a further object of the present invention to provide
2 a method to measure (or estimate) the shear modulus of a slab of
3 material subjected to insonification.

4 It is a still further object of the present invention to
5 provide a method to measure (or estimate) the Young's modulus of
6 a slab of material subjected to insonification.

7 It is a still further object of the present invention to
8 provide a method to measure (or estimate) the complex frequency-
9 dependent Poisson's ratio of a slab of material subjected to
10 insonification.

11 To attain the objects described, there is provided a method
12 which uses three transfer functions that are obtained by
13 insonifying the material at different angles. Once this is
14 accomplished, the transfer functions are manipulated with an
15 inverse method to yield closed form values of dilatational and
16 shear wavespeeds at any given test frequency. The wavespeeds are
17 combined to determine complex Lamé constants, complex Young's
18 modulus, complex shear modulus, and complex Poisson's ratio.

19

20 **BRIEF DESCRIPTION OF THE DRAWINGS**

21 A more complete understanding of the invention and many of
22 the attendant advantages thereto will be readily appreciated as
23 the same becomes better understood by reference to the following

1 detailed description when considered in conjunction with the
2 accompanying drawings wherein:

3 FIG. 1 depicts a test setup to insonify and gather
4 measurements for a specimen of material;

5 FIG. 2 depicts the coordinate system of the test setup of
6 FIG. 1;

7 FIG. 3 is a plotted graph depicting the measurable transfer
8 function of magnitude versus the frequency;

9 FIG. 4 is a plotted graph depicting the measurable transfer
10 function of phase angle versus the frequency;

11 FIG. 5 is a plotted graph depicting the measurable transfer
12 function "s" versus the frequency;

13 FIG. 6 is a plotted graph depicting the real component of
14 the actual and estimated dilatational wavespeed versus the
15 frequency;

16 FIG. 7 is a plotted graph depicting the imaginary component
17 of the actual and estimated dilatational wavespeed versus the
18 frequency;

19 FIG. 8 is a plotted graph depicting the surface defined in
20 equation (64) of the description versus the real and imaginary
21 components of β_2 at 1800 Hz with the magnitude depicted as a
22 gray scale image;

23 FIG. 9 is a plotted graph depicting the contour of the
24 surface versus both the real and imaginary parts of β_2 ;

1 FIG. 10 is a plotted graph depicting the actual shear
2 wavespeed and the estimated shear wavespeed versus the frequency
3 with the real component;

4 FIG. 11 is a plotted graph depicting the actual shear
5 wavespeed and the estimated shear wavespeed versus the frequency
6 with the imaginary component;

7 FIG. 12 is a plotted graph depicting the actual shear
8 modulus and the estimated shear modulus versus the frequency
9 with the real component;

10 FIG. 13 is a plotted graph depicting the actual shear
11 modulus and the estimated shear modulus versus the frequency
12 with the imaginary component;

13 FIG. 14 is a plotted graph depicting the actual Young's
14 modulus and the estimated Young's modulus versus the frequency
15 with the real component;

16 FIG. 15 is a plotted graph depicting the actual Young's
17 modulus and the estimated Young's modulus versus the frequency
18 with the imaginary component; and

19 FIG. 16 is a plotted graph depicting the actual Poisson's
20 ratio and the estimated Poisson's ratio versus the frequency.

DESCRIPTION OF THE PREFERRED EMBODIMENT

Referring now to the drawings wherein like numerals refer to like elements throughout the several views, one sees that FIG. 1 depicts the insonification of a slab-shaped test specimen 10 by a speaker (or projector) 12. Insonification consists of loading the specimen 10 on one entire side with an acoustic wave originating at the speaker 12. The speaker 12 is located at a sufficient distance from the specimen 10 that the acoustic wave is nearly a plane wave by the time it contacts the specimen. The insonification is usually done at multiple frequencies and multiple angles.

For the method presented, a frequency sweep (swept sine) is conducted at three different insonification angles. The transfer function data is collected with either accelerometers 16, 18 on both sides which record accelerations, or laser velocimeters 20, 22 shining on both sides which record velocities. In the swept sine mode, the transfer functions of acceleration divided by acceleration or velocity divided by velocity are both equal to displacement divided by displacement. The time domain data are Fourier transformed into the frequency domain and then recorded as complex transfer functions, typically using a spectrum analyzer (not shown).

1 The motion of the specimen 10 is governed by the equation

$$2 \quad \mu \nabla^2 \mathbb{u} + (\lambda + \mu) \nabla \nabla \circ \mathbb{u} = \rho \frac{\partial^2 \mathbb{u}}{\partial t^2} , \quad (1)$$

4
5 where λ and μ are the complex Lamé constants (N/m^2), ρ is the
6 density (kg/m^3), \circ denotes a vector dot product; \mathbf{u} is the
7 Cartesian coordinate displacement vector of the material and ∂
8 is the partial differential.

9 The coordinate system of the test configuration is shown in
10 FIG 2. Note that using this orientation results in $b = 0$ and a
11 having a value less than zero. The thickness h of the specimen
12 10 is a positive value. Equation (1) is manipulated by writing
13 the displacement vector \mathbf{u} as

$$14 \quad \mathbb{u} = \begin{Bmatrix} u_x(x, y, z, t) \\ u_y(x, y, z, t) \\ u_z(x, y, z, t) \end{Bmatrix} , \quad (2)$$

16 where x is the location along the specimen 10, y is the location
17 into the specimen 10, and z is the location normal to the
18 specimen 10 and t is time (s). The symbol ∇ is the gradient
19 vector differential operator written in three-dimensional
20 Cartesian coordinates as

$$\nabla = \frac{\partial}{\partial x} i_x + \frac{\partial}{\partial y} i_y + \frac{\partial}{\partial z} i_z , \quad (3)$$

with i_x denoting the unit vector in the x-direction, i_y denoting the unit vector in the y-direction, and i_z denoting the unit vector in the z-direction; ∇^2 is the three-dimensional Laplace operator operating on vector \mathbf{u} as

$$\nabla^2 \mathbf{u} = \nabla^2 u_x i_x + \nabla^2 u_y i_y + \nabla^2 u_z i_z , \quad (4)$$

with ∇^2 operating on scalar u as

$$\nabla^2 u_{x,y,z} = \nabla \cdot \nabla u_{x,y,z} = \frac{\partial^2 u_{x,y,z}}{\partial x^2} + \frac{\partial^2 u_{x,y,z}}{\partial y^2} + \frac{\partial^2 u_{x,y,z}}{\partial z^2} ; \quad (5)$$

and the term $\nabla \cdot \mathbf{u}$ is called the divergence and is equal to

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} . \quad (6)$$

The displacement vector \mathbf{u} is written as

$$\mathbf{u} = \nabla \phi + \nabla \times \bar{\psi} , \quad (7)$$

1 where ϕ is a dilatational scalar potential, \times denotes a vector
 2 cross product, and $\vec{\psi}$ is an equivoluminal vector potential
 3 expressed as

$$\vec{\psi} = \begin{Bmatrix} \psi_x(x, y, z, t) \\ \psi_y(x, y, z, t) \\ \psi_z(x, y, z, t) \end{Bmatrix}. \quad (8)$$

6
 7 The problem is formulated as a two-dimensional system, thus $y \equiv 0$,
 8 $u_y(x, y, z, t) \equiv 0$, and $\partial(\cdot)/\partial y \equiv 0$. Expanding equation (7) and breaking
 9 the displacement vector into its individual nonzero terms yields

$$u_x(x, z, t) = \frac{\partial \phi(x, z, t)}{\partial x} - \frac{\partial \psi_y(x, z, t)}{\partial z} \quad (9)$$

12
 13 and

$$u_z(x, z, t) = \frac{\partial \phi(x, z, t)}{\partial z} + \frac{\partial \psi_y(x, z, t)}{\partial x}. \quad (10)$$

16 Equations (9) and (10) are next inserted into equation (1),
 17 which results in

$$c_d^2 \nabla^2 \phi(x, z, t) = \frac{\partial^2 \phi(x, z, t)}{\partial t^2} \quad (11)$$

20 and

$$c_s^2 \nabla^2 \psi_y(x, z, t) = \frac{\partial^2 \psi_y(x, z, t)}{\partial t^2} \quad (12)$$

where equation (11) corresponds to the dilatational component and equation (12) corresponds to the shear component of the displacement field. Correspondingly, the constants c_d and c_s are the complex dilatational and shear wave speeds, respectively, and are determined by

$$c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (13)$$

and

$$c_s = \sqrt{\frac{\mu}{\rho}} \quad (14)$$

The relationship of the Lamé constants to the Young's and shear moduli is shown as

$$\lambda = \frac{Ev}{(1+\nu)(1-2\nu)} \quad (15)$$

1 and

2

$$\mu = G = \frac{E}{2(1+\nu)}, \quad (16)$$

4 where E is the complex Young's modulus (N/m^2), G is the complex
5 shearmodulus (N/m^2), and ν is the Poisson's ratio of the material
6 (dimensionless).

7 The conditions of infinite length and steady-state response
8 are now imposed, allowing the scalar and vector potential to be
9 written as

10

$$\phi(x, z, t) = \Phi(z) \exp(ikx) \exp(i\omega t) \quad (17)$$

12

13 and

14

$$\psi_y(x, z, t) = \Psi(z) \exp(ikx) \exp(i\omega t) \quad (18)$$

16

17 where i is the square root of -1 , ω is frequency (rad/s), and k
18 is wavenumber with respect to the x axis (rad/m), Φ is the
19 amplitude of the scalar potential ϕ as a function of z and Ψ is
20 the amplitude of the vector potential as a function of z .

21 Inserting equation (17) into equation (11) yields

22

$$\frac{d^2\Phi(z)}{dz^2} + \alpha^2\Phi(z) = 0 , \quad (19)$$

where α is the modified dilatational wavenumber and d is the differential operator and where

$$\alpha = \sqrt{k_d^2 - k^2} , \quad (20)$$

with k_d is the actual dilatational wave number and

$$k_d = \frac{\omega}{c_d} . \quad (21)$$

Inserting equation (18) into equation (12) yields

$$\frac{d^2\Psi(z)}{dz^2} + \beta^2\Psi(z) = 0 , \quad (22)$$

where

$$\beta = \sqrt{k_s^2 - k^2} , \quad (23)$$

1 with β as the modified shear wavenumber and

2

$$3 \quad k_s = \frac{\omega}{c_s} . \quad (24)$$

4

5 The solution to equation (19) is

6

$$7 \quad \Phi(z) = A(k, \omega) \exp(i\alpha z) + B(k, \omega) \exp(-i\alpha z) , \quad (25)$$

8 and the solution to equation (22) is

$$9 \quad \Psi(z) = C(k, \omega) \exp(i\beta z) + D(k, \omega) \exp(-i\beta z) , \quad (26)$$

10

11 where $A(k, \omega)$, $B(k, \omega)$, $C(k, \omega)$, and $D(k, \omega)$ are wave response
12 coefficients that are determined below. The displacements can
13 now be written as functions of the unknown constants using the
14 expressions in equations (9) and (10). They are

15

$$16 \quad u_z(x, z, t) = U_z(k, z, \omega) \exp(ikx) \exp(i\omega t) \\ = \{ i\alpha [A(k, \omega) \exp(i\alpha z) - B(k, \omega) \exp(-i\alpha z)] + \\ ik [C(k, \omega) \exp(i\beta z) + D(k, \omega) \exp(-i\beta z)] \} \exp(ikx) \exp(i\omega t) , \quad (27)$$

17

18 with U_z as the amplitude of displacement in the "z" direction and

19

$$u_x(x, z, t) = U_x(k, z, \omega) \exp(ikx) \exp(i\omega t)$$

(28)

$$= \{ ik[A(k, \omega) \exp(iaz) + B(k, \omega) \exp(-iaz)] - \\ i\beta[C(k, \omega) \exp(i\beta z) - D(k, \omega) \exp(-i\beta z)] \} \exp(ikx) \exp(i\omega t)$$

with U_x as the amplitude of displacement in the "x" direction.

The normal stress at the top of the plate ($z = b$) is equal to

opposite the pressure load created by the projector. This

expression is

$$\tau_{zz}(x, b, t) = (\lambda + 2\mu) \frac{\partial u_z(x, b, t)}{\partial z} + \lambda \frac{\partial u_x(x, b, t)}{\partial x} = -p_0(x, b, t), \quad (29)$$

and the tangential stress at the top of the plate b is zero and

this equation is written as

$$\tau_{zx}(x, b, t) = \mu \left[\frac{\partial u_x(x, b, t)}{\partial z} + \frac{\partial u_z(x, b, t)}{\partial x} \right] = 0. \quad (30)$$

The normal stress the bottom of the plate ($z = a$) is equal to

zero. This expression is

$$\tau_{zz}(x, a, t) = (\lambda + 2\mu) \frac{\partial u_z(x, a, t)}{\partial z} + \lambda \frac{\partial u_x(x, a, t)}{\partial x} = 0, \quad (31)$$

1 and the tangential stress at the bottom of the plate is zero and
2 this equation is written as

3

$$4 \quad \tau_{zx}(x,a,t) = \mu \left[\frac{\partial u_x(x,a,t)}{\partial z} + \frac{\partial u_z(x,a,t)}{\partial x} \right] = 0 . \quad (32)$$

5 The applied load in equation (29) is an acoustic pressure
6 and is modeled as a function at definite wavenumber and
7 frequency as

8

$$9 \quad p_0(x,z,t) = P_0(\omega) \exp(ikx) \exp(i\omega t) , \quad (33)$$

10

11 with P being the amplitude and where the wavenumber k is found
12 using

13

$$14 \quad k = \frac{\omega}{c_f} \sin(\theta) , \quad (34)$$

15

16 where c_f is the compressional wavespeed of air (m/s) and θ is the
17 angle of incidence of the projector with the z axis (rad).

18 Assembling equations (1) - (34) and letting $b = 0$ yields
19 the "A" matrix, x vector, and b vector in a four-by-four system
20 of linear equations that model the system written in matrix
21 form. They are

22

$$\mathbf{Ax} = \mathbf{b} ; (\mathbf{A} \text{ } 4 \times 4, \mathbf{x} \text{ } 4 \times 1, \mathbf{b} \text{ } 4 \times 1) \quad (35)$$

where the entries of equation (35) are

$$A_{11} = -\alpha^2 \lambda - 2\alpha^2 \mu - \lambda k^2 , \quad (36)$$

$$A_{12} = A_{11} , \quad (37)$$

$$A_{13} = 2k\beta\mu , \quad (38)$$

$$A_{14} = A_{13} , \quad (39)$$

$$A_{21} = -2\mu k\alpha , \quad (40)$$

$$A_{22} = -A_{21} , \quad (41)$$

$$A_{23} = \mu\beta^2 - \mu k^2 , \quad (42)$$

$$A_{24} = A_{23} , \quad (43)$$

$$A_{31} = A_{11} \exp(i\alpha a) , \quad (44)$$

$$1 \quad A_{32} = A_{11} \exp(-i\alpha a) , \quad (45)$$

$$2 \quad$$

$$3 \quad A_{33} = -A_{13} \exp(i\beta a) , \quad (46)$$

$$4 \quad A_{34} = A_{13} \exp(-i\beta a) , \quad (47)$$

$$5 \quad$$

$$6 \quad A_{41} = A_{21} \exp(i\alpha a) , \quad (48)$$

$$7 \quad$$

$$8 \quad A_{42} = -A_{21} \exp(-i\alpha a) , \quad (49)$$

$$9 \quad$$

$$10 \quad A_{43} = A_{23} \exp(i\beta a) , \quad (50)$$

$$11 \quad$$

$$12 \quad A_{44} = A_{23} \exp(-i\beta a) , \quad (51)$$

$$13 \quad$$

$$14 \quad b_{11} = -P_0(\omega) , \quad (52)$$

$$15 \quad$$

$$16 \quad b_{21} = 0 , \quad (53)$$

$$17 \quad$$

$$18 \quad b_{31} = 0 , \quad (54)$$

19 and

$$20 \quad$$

$$21 \quad b_{41} = 0 . \quad (55)$$

Using equations (35) - (55) the solution to the constants $A(k, \omega)$, $B(k, \omega)$, $C(k, \omega)$, and $D(k, \omega)$ can be calculated at each specific wavenumber and frequency using

$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \quad (56)$$

Once these are known, the transfer function T between the wall motion in the z direction at $z = a$ and the wall motion in the z direction at $z = b$ is now written in closed form notation using equations (27) and (56). The resulting expression is

$$T(k, \omega) = \frac{U_z(k, a, \omega)}{U_z(k, b, \omega)} = \frac{4\alpha\beta k^2 \sin(\alpha h) + (\beta^2 - k^2)^2 \sin(\beta h)}{4\alpha\beta k^2 \sin(\alpha h) \cos(\beta h) + (\beta^2 - k^2)^2 \cos(\alpha h) \sin(\beta h)} \quad (57)$$

The first step is to solve for the response at zero wavenumber, or what is typically referred to as broadside response, to determine the dilatational wavespeed. At zero wavenumber, the angle between the direction of propagation of the insonification energy and the z axis is zero. The response of the structure to broadside energy is comprised entirely of dilatational waves, i.e., no shear waves are excited at zero wavenumber. Furthermore, the transfer function given in equation (57) reduces to

$$T(0, \omega) = \frac{1}{\cos(\alpha_1 h)} = T_1 = \frac{1}{R_1}, \quad (58)$$

where T_1 (or R_1) is the measurement data from the experiment with a insonification angle of zero and is typically a frequency-dependent complex number and the subscript 1 denotes the first experiment. Equation (58) can be expanded into real and imaginary parts and solved, resulting in a value for α_1 at every frequency in which a measurement is made. The solution to the real part Re of α_1 is

$$\text{Re}(\alpha_1) = \begin{cases} \frac{1}{2h} \text{Arc cos}(s) + \frac{n\pi}{2h} & n \text{ even} \\ \frac{1}{2h} \text{Arc cos}(-s) + \frac{n\pi}{2h} & n \text{ odd,} \end{cases} \quad (59)$$

where

$$s = [\text{Re}(R_1)]^2 + [\text{Im}(R_1)]^2 - \sqrt{\{[\text{Re}(R_1)]^2 + [\text{Im}(R_1)]^2\}^2 - \{2[\text{Re}(R_1)]^2 - 2[\text{Im}(R_1)]^2 - 1\}}, \quad (60)$$

and n is a non-negative integer and the capital A denotes the principal value of the inverse cosine function. The value of n is determined from the function s , which is a periodically varying cosine function with respect to frequency. At zero

1 frequency, n is 0. Every time s cycles through π radians (180
 2 degrees), n is increased by 1. When the solution to the real
 3 part of α_1 is found, the solution to the imaginary part Im of α_1
 4 is then written as

$$\text{Im}(\alpha_1) = \frac{1}{h} \log_e \left\{ \frac{\text{Re}(R_1)}{\cos[\text{Re}(\alpha_1)h]} - \frac{\text{Im}(R_1)}{\sin[\text{Re}(\alpha_1)h]} \right\}. \quad (61)$$

7
 8 The real and imaginary parts of α_1 from equations (59) and (61)
 9 respectively are combined to yield the complex wavenumber.
 10 Because this measurement is made at zero wavenumber ($k \equiv 0$), this
 11 is equal to the dilatational wavenumber. Thus, the dilatational
 12 wavespeed is equal to

$$c_d = \frac{\omega}{[\text{Re}(\alpha_1) + i\text{Im}(\alpha_1)]}. \quad (62)$$

15
 16 To solve for the shear wavespeed, the specimen must be excited
 17 at a nonzero wavenumber. This is done next.

18 The next step is to solve for the response at nonzero
 19 wavenumber to determine the shear wavespeed. At nonzero
 20 wavenumber, the transfer function is given in equation (57).
 21 For this nonzero angle of insonification, this can be expressed
 22 as

$$T(k, \omega) = \frac{4\alpha_2\beta_2k_2^2 \sin(\alpha_2h) + (\beta_2^2 - k_2^2)^2 \sin(\beta_2h)}{4\alpha_2\beta_2k_2^2 \sin(\alpha_2h)\cos(\beta_2h) + (\beta_2^2 - k_2^2)^2 \cos(\alpha_2h)\sin(\beta_2h)} = T_2 = \frac{1}{R_2}, \quad (63)$$

where T_2 (or R_2) is the measurement data from the experiment at nonzero insonification angle and is typically a frequency-dependent complex number and the subscript 2 denotes the second angle or experiment. It is noted that α_2 in equation (63) is different from α_1 in equation (58). This difference is based on a k^2 term shown in equation (20) where the wavenumber α is defined. Due to the complexity of equation (63), there is no simple method to rewrite the equation as a function f of β_2 , the variable that is to be estimated. Equation (63) can be rewritten as

$$f(\beta_2) = 0 = 4\alpha_2\beta_2k_2^2 \sin(\alpha_2h)[\cos(\beta_2h) - R_2] + (\beta_2^2 - k_2^2)^2 \sin(\beta_2h)[\cos(\alpha_2h) - R_2], \quad (64)$$

where the problem now becomes finding the zeros of the right hand side of equation (64), or in the presence of actual data that contains noise, finding the relative minima of the right hand side of equation (64) and determining which relative minimum corresponds to shear wave propagation and which relative minima are extraneous. Because equation (64) has a number of relative minima, zero finding algorithms are not applied to this

function, as they typically do not find all of the minima locations. The best method to find all of the minima locations is by plotting the absolute value of the right hand side of equation (64) as a surface with the real part of β_2 on one axis and the imaginary part of β_2 on the other axis. The value α_2 is determined using

$$\alpha_2 = \sqrt{k_d^2 - k_2^2} = \sqrt{\alpha_1^2 - k_2^2} , \quad (65)$$

so that equation (64) is a function of only β_2 . Once this function is plotted, the minima can be easily identified and the corresponding value of $(\beta_2)_m$ at the location of the minima can be determined by examination of the minimum location, sometimes referred to as the grid method. The shear wave speed(s) are then determined using

$$(k_s)_m = \sqrt{(\beta_2)_m^2 + k_2^2} \quad (66)$$

and

$$(c_s)_m = \frac{\omega}{(k_s)_m} \quad (67)$$

1 where the subscript m denotes each minima value that was found
2 by inspecting the surface formed from equation (64). The
3 determination of the correct index of m that corresponds to
4 shear wave propagation is done below.

5 The material properties such as Young's modulus and other
6 material properties can be determined from the wavespeeds. The
7 Lamé constants are calculated with equations (13) and (14)
8 written as

$$10 \quad \mu_m = \rho(c_s)_m^2 \quad (68)$$

11
12 and

$$14 \quad \lambda_m = \rho c_d^2 - 2\rho(c_s)_m^2 \quad (69)$$

15
16 To determine the correct index m that corresponds to the actual
17 wave propagation rather than an extraneous solution, a third set
18 of measurements are made at a nonzero incidence angle that is
19 not equal to the angle used previously. The model in equation
20 (63) is calculated from the estimated material properties and a
21 residual value is determined using the third set of
22 measurements. Each m indexed residual at a specific frequency
23 is defined as

$$(\varepsilon_3)_m = \frac{4\alpha_3(\beta_3)_m k_3^2 \sin(\alpha_3 h) + [(\beta_3)_m^2 - k_3^2]^2 \sin[(\beta_3)_m h]}{4\alpha_3(\beta_3)_m k_3^2 \sin(\alpha_3 h) \cos[(\beta_3)_m h] + [(\beta_3)_m^2 - k_3^2]^2 \cos(\alpha_3 h) \sin[(\beta_3)_m h]} - \frac{1}{R_3}, \quad (70)$$

where

$$\alpha_3 = \sqrt{k_d^2 - k_3^2} = \sqrt{\alpha_1^2 - k_3^2} \quad (71)$$

and

$$(\beta_3)_m = \sqrt{(\beta_2)_m^2 + k_2^2 - k_3^2}, \quad (72)$$

and the subscript 3 denotes the third experiment. The smallest residual value corresponds to the correct value of index m and the correct values of Lamé constants. Poisson's ratio is then calculated using

$$\nu = \frac{\lambda}{2(\mu + \lambda)}. \quad (73)$$

Young's modulus can be calculated with

$$E = \frac{2\mu(2\mu + 3\lambda)}{2(\mu + \lambda)} \quad (74)$$

1 and the shear modulus can be determined using

$$G \equiv \mu . \quad (75)$$

2
3
4
5 The above measurement method can be simulated by means of a
6 numerical example. Soft rubber-like material properties are
7 used in this simulation. The material has a Young's modulus E of
8 $[(1e8 - i2e7) + (5e3f - i3e2f)] \text{ N/m}^2$ where f is frequency in Hz, Poisson's
9 ratio ν is equal to 0.40 (dimensionless), density ρ is equal to
10 1200 kg/m^3 , and a thickness h of 0.01 m. A compressional
11 (acoustic) wave velocity of c_f of 343 m/s for air is used. All
12 other parameters can be calculated from these values. The
13 insonification angles of zero, twenty, and forty degrees are
14 chosen to illustrate this method.

15 FIGS. 3 and 4 are plots of the transfer functions of
16 equation (57) at zero (x symbol), twenty (o symbol), and forty
17 degree (+ symbol) insonification angles versus the frequency.
18 FIG. 3 represents the magnitude of the transfer function versus
19 the frequency and FIG. 4 represents the phase angle versus the
20 frequency.

21 Once the transfer functions are known (typically by
22 measurement but here by numerical simulation), the dilatational
23 wavespeed can be estimated using equations (59) - (62). FIG. 5

1 is a plot of the function s versus the frequency. FIGS. 6 and 7
2 are plots of the actual dilatational wavespeed (solid line) and
3 the estimated dilatational wavespeed (o symbol) versus the
4 frequency. FIG. 6 depicts the real component and FIG. 7 depicts
5 the imaginary component.

6 FIGS. 8 and 9 are plots of the surface defined in equation
7 (64) versus real and imaginary components of β_2 at 1800 Hz.
8 FIG. 8 depicts a gray scale image of the magnitude versus the
9 real part of β_2 and FIG. 9 depicts a contour plot of the surface
10 versus both the real and imaginary parts of β_2 . For both
11 figures, there are six distinct local minima that are labeled in
12 bold numbers. The seventh local minima corresponds to $\beta_2 = 0$
13 which implies there is no shear wave propagation; a physically
14 unrealizable condition at nonzero wavenumber. These six local
15 minima are processed at a third measurement location according
16 to equation (70). The results are listed in Table 1. Local
17 minimum number 3 has the smallest residual value and corresponds
18 to the shear wave propagation. The value for $(\beta_2)_3$ is equal to
19 $61.3 + 5.9i$ compared to the actual value of β_2 which is $61.0 +$
20 $5.9i$. The small difference between the two values can be
21 attributed to discretization of the surface shown in FIG. 9.

1 Table 1. Values of $(\beta_2)_m$ and $(\epsilon_3)_m$ at the Local Minima

2

Local Minima Number m	Value of $(\beta_2)_m$	Residual $(\epsilon_3)_m$ (Equation (70))
1	22.6 + 5.2i	0.257
2	38.8 + 2.5i	2.064
3	61.3 + 5.9i	0.013
4	94.5 + 1.1i	0.426
5	125.1 + 1.0i	0.326
6	157.5 + 1.1i	0.349

3

4 FIGS. 10 and 11 are plots of the actual shear wavespeed
5 (solid line) and the estimated shear wavespeed (o symbol) versus
6 the frequency. FIG. 10 depicts the real component and FIG. 11
7 depicts the imaginary component. As in FIGS. 8 and 9, the small
8 difference between the two values can be attributed to
9 discretization of the surface.

10 FIGS. 12 and 13 are plots of the actual shear modulus
11 (solid line) and the estimated shear modulus (o symbol) versus
12 the frequency. FIG. 12 depicts the real component and FIG. 13
13 depicts the imaginary component.

FIGS. 14 and 15 are plots of the actual Young's modulus (solid line) and the estimated Young's modulus (o symbol) versus the frequency. FIG. 14 depicts the real component and FIG. 15 depicts the imaginary component. Finally, FIG. 16 is a plot of the actual Poisson's ratio (solid line) and the estimated Poisson's ratio (o symbol) versus frequency. Because the numerical example is formulated using a Poisson's ratio that is strictly real, no imaginary component is shown in this plot. Imaginary values of Poisson's ratio are possible and have been shown to theoretically exist (See T. Pritz, "Frequency Dependencies of Complex Moduli and Complex Poisson's Ratio or Real Solid Materials," *Journal of Sound and Vibration*, Volume 214, Number 1, 1998, pp. 83-104).

The major advantages of this new method is the ability to estimate complex dilatational and shear wavespeeds of a material that is slab-shaped and subjected to insonification; the ability to estimate complex Lamé constants of the material; the ability to estimate complex Young's and shear moduli of the material and the ability to estimate complex Poisson's ratio of the material.

Thus by the present invention its objects and advantages are realized and although preferred embodiments have been disclosed and described in detail herein, its scope should be determined by that of the appended claims.